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COMPLETE NONORIENTABLE MINIMAL SURFACES IN R^3

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Dedicated to Professor Tadashi Nagano on his 60th birthday

ABSTRACT. We will study complete minimal immersions of nonorientable surfaces into R^3. Especially, we construct a nonorientable surface P_2 which is homeomorphic to a Klein bottle and show that for any integer $m \geq 4$, there are complete minimal immersion of $M = P_2 - \{q\}$, $q \in P_2$ in R^3 with one end and total curvature $C(M) = -4m\pi$.

1. INTRODUCTION

At first, we will construct nonorientable surfaces of genus n $(n \geq 1)$. Let T_{n-1} be a hyperelliptic Riemann surface given by

$$T_{n-1} = \left\{ (z, w) \in (C \cup \{\infty\})^2, \, w^2 = \prod_{i=1}^{n} (a_i - z)(\overline{a}_i + z) \right\},$$

where, $a_i \neq a_j$ for any $i \neq j$, and $a_i \neq -\overline{a}_j$ for any i, j. Define a map $I: T_{n-1} \to T_{n-1}$ by $I(z, w) = (-\overline{z}, -\overline{w})$. Then, the map I is an antiholmorphic involution. Moreover it has no fixed point. In fact, let (z, w) be a fixed point of I, that is, $z = ib$ and $w = ic$, where b and c are real numbers. Then, since $|a_i| > \operatorname{Im}(a_i)$, we have a contradiction:

$$-c^2 = \prod (|a_i|^2 + b^2 - 2\operatorname{Im}(a_i)b) > 0.$$

Let P_n be the quotient space of T_{n-1} by the equivalence relation defined by $(z_1, w_1) \sim (z_2, w_2) \Leftrightarrow (z_2, w_2) = I(z_1, w_1)$. Then the canonical projection $\pi: T_{n-1} \to P_n$ is the two-sheeted covering and P_n is a nonorientable surface of genus n. We may consider P_1 as the projective plane and P_2 as the Klein bottle.

In the present paper, we will study minimal immersions of $M = P_n - \{q_1, \dots, q_r\}$ into R^3, where $q_1, \dots, q_r \in P_n$. Then we have the orientable double covering $\pi: \widetilde{M} \to M$, where $\widetilde{M} = T_{n-1} - \{p_1, I(p_1), \dots, p_r, I(p_r)\}$ and $q_1 = \pi(p_1), \dots, q_r = \pi(p_r)$. It is shown in [7] that for a complete minimal immersion $\tilde{x}: \widetilde{M} \to R^3$, there is a complete minimal immersion $x: M \to R^3$ if and only if $\tilde{x}(I(p)) = \tilde{x}(p)$ for all $p \in \widetilde{M}$. In this case, $\tilde{x}: \widetilde{M} \to R^3$ is called the double surface of $x: M \to R^3$. The points q_i's and also p_i's, $I(p_i)$'s

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are called the endpoints of the immersions. We need the following representation theorem of Meeks [7] and Oliveira [8] which is followed from the classical Enneper-Weierstrass representation theorem.

Proposition 1.1 (Meeks [7] and Oliveira [8]). *Let* $g: \widetilde{M} \to C \cup \{\infty\}$ *be a meromorphic function and* η *be a holomorphic* 1*-form on* $\widetilde{M}$. *Take the vector-valued* 1*-form*

$$\Phi = (\phi_1, \phi_2, \phi_3) = ((1-g^2)\eta, i(1+g^2)\eta, 2g\eta).$$

Assume that no component of Φ *has a real peroid and that the poles of* g *coincide with zeros of* η *and the order of a pole of* g *is precisely half the order of a zero of* η. *Then* $\tilde{x}(p) = \operatorname{Re}\int_{p_0}^{p}\Phi$ *is a conformal regular minimal immersion. Moreover* $\tilde{x}(I(p)) = \tilde{x}(p)$ *for all* $p \in \widetilde{M}$ *if and only if* $I^*(\Phi) = \overline{\Phi}$, *in other words*

$$g(I(p)) = -1/\overline{g(p)}, \quad I^*(\eta) = -\overline{g^2\eta}, \quad \text{for } p \in M.$$

In this case, there is a regular minimal immersion $x: M \to R^3$ *such that* $\tilde{x} = x \cdot \pi$.

Conversely, every regular minimal immersion $x: M \to R^3$ *can be given in this way.*

The meromorphic function g in the above is the stereographic projection of the Gauss map of $\tilde{x}$. In the case of finite total curvature, it is known from the Osserman's result [9], g is extended to a meromorphic function on T_{n-1}. In the present paper, we will study only minimal immersions with finite total curvature. We will call (g, η) in the above the Weierstrass pair of x.

Particularly, we are concerned with elliptic Riemann surfaces and corresponding nonorientable surfaces of genus 2 with one end. Put

$$T_1 = \{(z, w) \in (C \cup \{\infty\})^2; w^2 = (1-z^2)(1-k^2z^2)\}, \qquad 0 < k < 1.$$

Then T_1 is homeomorphic to a torus. The quotient space P_2 is homeomorphic to a Klein bottle. As for any two points p_1, p_2 of T_1, we have a conformal transformation which maps p_1 to p_2, we may assume that the endpoint is $\pi(\infty, \infty)$, where $\pi: T_1 \to P_2$ is the natural projection. We will study complete minimal immersions of $M = P_2 - \{\pi(\infty, \infty)\}$. Our main result is the following

Theorem. *Let* M *be a nonorientable surface* $P_2 - \{q\}$ *with one end* q. *Then*

(1) *there is no complete minimal immersion of* M *into* R^3 *whose total curvature* $C(M)$ *is greater than* -14π,

(2) *for any* $0 < k < 1$ *and any integer* $m \geq 4$, *there are complete minimal immersions of* M *into* R^3 *with* $C(M) = -4m\pi$.

2. A Weierstrass pair for a nonorientable surface

In this section, we will determine the forms of g and η introduced in the previous section. Since g is a meromorphic function on the hyperelliptic surface T_{n-1}, it is represented as

$$g(z, w) = (Rw + P)/Qw, \quad (z, w) \in T_{n-1}, \tag{2.1}$$

where R, P and Q are polynomials of z. Put $\eta = f(z, w)dz$. Then from Proposition 1.1 we have

$$\overline{f(I(z, w))} = g(z, w)^2 f(z, w). \tag{2.2}$$

Taking account of the conditions for a Weierstrass pair, we can put

$$f(z, w) = (Q^2w^2/S) \times (1/w), \tag{2.3}$$

where S is a polynomial of z which vanishes only at the endpoints. We need the last factor $1/w$, because the corresponding immersion is regular at points $(a_i, w(a_i))$ and $I(a_i, w(a_i))$, $1 \le i \le n$. Hence if these all points are endpoints, we may drop the factor $1/w$. Substituting (2.1) and (2.3) into (2.2), we have

$$S\widetilde{Q}^2w^2 + \widetilde{S}Q^2(R^2w^2 + P^2) = -2\widetilde{S}Q^2PRw,$$

where $\widetilde{Q} = \overline{Q(-\overline{z})}$ and $\widetilde{S} = \overline{S(-\overline{z})}$. The left-hand side of the above equation depends on z only. On the other hand, its right-hand side depends on z and w. Hence they must vanish. As $S \neq 0$ and $Q \neq 0$, we have $P = 0$ or $R = 0$. At first, we assume $P = 0$. Then we can put

$$g = A\prod_{i=1}^{m}(z - d_i)/\prod_{j=1}^{s}(z - c_j), \qquad (c_i \neq d_j \text{ for all } i, j).$$

Since g satisfies $g(I(p)) = -1/\overline{g(p)}$, it follows

$$|A|^2(-1)^{s-m}\prod_{i=1}^{m}(z - d_i)(z + \overline{d}_i) = -\prod_{j=1}^{s}(z - c_j)(z + \overline{c}_j).$$

Hence it must be $s = m$. But then we have $|A|^2 = -1$. Thus, we get $R = 0$ and $P \neq 0$. Now we can set

$$g(z, w) = A\prod_{i=1}^{t}(z - b_i)/\left(w\prod_{j=1}^{m}(z - c_j)\right), \qquad (b_i \neq c_j \text{ for all } i, j). \tag{2.4}$$

From the condition about g, it follows

$$\begin{aligned}(-1)^{t-m-n}|A|^2&\prod_{j=1}^{t}(z - b_j)(z + \overline{b}_j)\\ &= \prod_{k=1}^{m}(z - c_k)(z + \overline{c}_k)\prod_{i=1}^{n}(z - a_i)(z + \overline{a}_i).\end{aligned}$$

Hence we get $t = m + n$ and $|A|^2 = 1$. Since $c_i \neq b_j$, if it is necessary, permuting the order of b_i's and replacing a_i by $-\overline{a}_i$, we may put $c_i = -\overline{b}_i$ for $1 \le i \le m$, $b_{m+j} = a_j$ for $1 \le j \le n$. Thus we obtain

$$g(z, w) = A\prod_{j=1}^{n}(z - b_j)\prod_{i=1}^{n}(z - a_i)/w\prod_{j=1}^{m}(z + \overline{b}_j), \qquad |A| = 1. \tag{2.5}$$

Let $p_1 = (c_1, w_1), I(p_1), \ldots, p_r = (c_r, w_r), I(p_r)$ be the endpoints. If they contain the infinite point (∞, ∞), we may assume $p_r = (\infty, \infty)$. We put $r^* = r - 1$ or r according to $p_r = (\infty, \infty)$ or not. Now we will determine the form of $\eta = f dz$. At first, we assume $c_i \neq a_j$, $c_i \neq -\overline{a}_j$ for all i, j. As g is given by (2.5), we can put

$$f = B\prod_{j=1}^{m}(z + \overline{b}_j)^2\prod_{i=1}^{n}(z + \overline{a}_i)/w\prod_{k=1}^{r^*}(z - c_k)^{\alpha_k}(z + \overline{c}_k)^{\beta_k}.$$

Using (2.2), we get

$$(-1)^{\alpha_1+\beta_1+\cdots+\alpha_{r^*}+\beta_{r^*}+1}\overline{B}/\prod_{k=1}^{r^*}(z+\overline{c}_k)^{\alpha_k}(z-\overline{c}_k)^{\beta_k}$$
$$= A^2B/\prod_{k=1}^{r^*}(z-c_k)^{\alpha_k}(z+\overline{c}_k)^{\beta_k}.$$

Hence we have $\alpha_k = \beta_k$ when $c_k \neq -\overline{c}_k$. We may assume that $c_k \neq -\overline{c}_k$ for $1 \leq k \leq p$ and $c_k = -\overline{c}_k$ for $p+1 \leq k \leq r^*$. Now we can modify f as

$$(2.6)\quad f = B\prod_{j=1}^{m}(z+\overline{b}_j)^2\prod_{i=1}^{n}(z+\overline{a}_i)/\left(w\prod_{k=1}^{p}(z-c_k)^{\alpha_k}(z+\overline{c}_k)^{\alpha_k}\prod_{j=1}^{q}(z-d_j)^{\beta_j}\right),$$

where $p+q = r^*$, the d_j's are zero or pure imaginary numbers and

$$(2.7)\quad \begin{aligned} &\overline{AB} = -(-1)^{\beta}AB, \quad \beta = \beta_1+\cdots+\beta_q, \\ &\qquad\qquad \alpha_k \geq 2, \ \beta_j \geq 2 \quad \text{for } 1 \leq k \leq p, \ 1 \leq j \leq q. \end{aligned}$$

The above inequalities follow from the fact that the orders of the poles are greater than 1 (see [2, 8]).

When some of c_i's coincide with some of a_i's, we can show after all that f has the same form as (2.6). In order to study the behavior of η at (∞, ∞), we put $z = 1/\xi$. Then we have, by putting $\alpha = \alpha_1+\cdots+\alpha_p$,

$$\eta = (\xi)^{2\alpha+\beta-2(m+1)}B\prod_{j=1}^{m}(1+\overline{b}_j\xi)^2$$
$$\times\prod_{i=1}^{n}(1+\overline{a}_i\xi)/\left(w\prod_{k=1}^{p}(1-c_k\xi)^{\alpha_k}(1+\overline{c}_k\xi)^{\alpha_k}\prod_{j=1}^{q}(1-d_j\xi)^{\beta_j}d\xi\right).$$

Thus, if $p_r = (\infty, \infty)$, it is a pole of order ≥ 2. Hence we have $2\alpha+\beta \leq 2m$. On the other hand, if all endpoints are finite, as an infinite point (∞, ∞) is neither a pole nor a zero, we get $2\alpha+\beta = 2(m+1)$. Now we obtain

Lemma 2.1. *A Weierstrass pair $(g, \eta = f\,dz)$ on $\widetilde{M}$ which constructs a nonorientable complete minimal immersion has the form* (2.5) *and* (2.6). *If one of the endpoints is an infinite point (∞, ∞), we have $2\alpha+\beta \leq 2m$. If all endpoints are finite points, we have $2\alpha+\beta = 2(m+1)$.*

3. The conditions under which Φ has no real period

The vector-valued 1-form Φ for a Weierstrass pair given by (2.5) and (2.6) is given by

$$(3.1)\quad \Phi = (B\psi_1+\overline{B}\psi_2,\ i(B\psi_1-\overline{B}\psi_2),\ (-1)^n 2AB\psi_3), \quad \overline{AB} = (-1)^{\beta+1}AB,$$

where

$$\psi_1 = \left(\prod_{j=1}^{m}(z+\overline{b}_j)^2\prod_{j=1}^{n}(z+\overline{a}_j)/(Rw)\right)dz,$$

$$\psi_2 = (-1)^{n+\beta}\left(\prod_{j=1}^{m}(z-b_j)^2\prod_{j=1}^{n}(z-a_j)/(Rw)\right)dz,$$

$$\psi_3 = \left(\prod_{j=1}^{m}(z-b_j)(z+\overline{b}_j)/R\right)dz,$$

$$R = \prod_{i=1}^{p}((z-c_i)(z+\overline{c}_i))^{\alpha_i}\prod_{j=1}^{q}(z-d_j)^{\beta_j}.$$

We must determine b_j such that Φ has no real period. The existence of real periods must be searched among the cycles that generate the fundamental group of $\widetilde{M}$. These are the ones that generate the fundamental group of T_{n-1} and the ones around the endpoints, $p_1, \dots, p_r$. Let $\widetilde{\beta}_1(t)$, $0 \le t \le 1$, be a curve connecting $-\overline{a}_1$ to a_1 and $\widetilde{\beta}_i(t)$, $0 \le t \le 1$, be curves connecting a_{i-1} to a_i for $2 \le i \le n$. Let

$$\gamma_i(t) = \begin{cases} \beta_i(t) = (\widetilde{\beta}_i(2t), w(\widetilde{\beta}_i(2t))), & \text{for } 0 \le t \le 1/2, \\ \beta_i^*(t) = (\widetilde{\beta}_i(2-2t), -w(\widetilde{\beta}_i(2-2t))), & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then, $\gamma_1, \dots, \gamma_n, I(\gamma_2), \dots, I(\gamma_{n-1})$ generate the fundamental group of T_{n-1}. The condition (3.1) implies $\int_{I(\gamma_j)}\phi_i = \int_{\gamma_j} I^* \ (\phi_i) = \int_{\gamma_j}\overline{\phi}_i$, for $1 \le i \le 3$. As ϕ_i contain the factor $1/w$ for $i = 1, 2$, we have

$$\int_{\gamma_j}\phi_i = \int_{\beta_j}\phi_i + \int_{\beta_j^*}\phi_i = 2\int_{\beta_j}\phi_i, \quad \text{for } i = 1, 2.$$

Similarly we have $\int_{\gamma_j}\phi_3 = \int_{\beta_j}\phi_3 + \int_{\beta_j^*}\phi_3 = \int_{\beta_j}\phi_3 - \int_{\beta_j}\phi_3 = 0$. Hence Φ has no real period on the fundamental cycles if and only if

$$\operatorname{Re}\left(\int_{\beta_j}\phi_i\right) = 0 \quad \text{for } i = 1, 2 \text{ and } 1 \le j \le n,$$

which are expressed by

$$\operatorname{Re}\left(\int_{\beta_j} B\psi_1\right) - \operatorname{Re}\left(\int_{\beta_j}\overline{B}\psi_2\right) = 0,$$

$$\operatorname{Im}\left(\int_{\beta_j} B\psi_1\right) - \operatorname{Im}\left(\int_{\beta_j}\overline{B}\psi_2\right) = 0.$$

Combining them together, we obtain, as we have $\overline{\psi}_2 = I^*(\psi_1)$,

$$\int_{\beta_j}\psi_1 + \int_{\beta_j} I^*(\psi_1) = 0. \tag{3.2}$$

Put

$$\psi_1 = \left(\sum_{j=0}^{l} A_j z^j + \sum_{i=1}^{p} \sum_{j=1}^{\alpha_i} (B_{ij}/(z-c_i)^j + C_{ij}/(z+\bar{c}_i)^j) \right. \tag{3.3}$$
$$\left. + \sum_{i=1}^{q} \sum_{j=1}^{\beta_i} D_{ij}/(z-d_i)^j \right) dz/w,$$

where $l = 2m + n - 2\alpha - \beta$. Now, from (3.2), we get

Lemma 3.1. *Put*

$$\int_{\beta_i} (z^j/w)dz = I_i^j, \quad \int_{\beta_i} (1/(z-c)^j w)dz = J_i^j(c).$$

Then the vector-valued 1*-form* Φ *has no real period along the fundamental cycles of* T_{n-1} *if and only if, for* $1 \le i \le n$,

$$\sum_{j=0}^{l} (I_i^j + (-1)^j \bar{I}_i^j) A_j + \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} (J_i^j(c_k) + (-1)^j \overline{J_i^j(-\bar{c}_k)}) B_{kj}$$
$$+ (J_i^j(-\bar{c}_k) + (-1)^j \overline{J_i^j(c_k)}) C_{kj}$$
$$+ \sum_{k=1}^{q} \sum_{j=1}^{\beta_k} (J_i^j(d_k) + (-1)^j \overline{J_i^j(d_k)}) D_{kj} = 0.$$

Next, we will study periods around the endpoints. As $I^*(\psi_3) = (-1)^{\beta+1}\overline{\psi_3}$, we can put

$$\prod_{j=1}^{m} (z-b_j)(z+\bar{b}_j)/R$$
$$= \sum_{j=0}^{l_1} E_j z^j + \sum_{k=1}^{p} \sum_{j=1}^{\alpha_k} (F_{kj}/(z-c_k)^j + (-1)^{\beta+j} \overline{F}_{kj}/(z+\bar{c}_k)^j) \tag{3.4}$$
$$+ \sum_{k=1}^{q} \sum_{j=1}^{\beta_k} G_{kj}/(z-d_k)^j,$$

where $\overline{G}_{kj} = (-1)^{\beta+j} G_{kj}$, $\overline{E}_j = (-1)^{\beta+j} E_j$, $l_1 = 2m - 2\alpha - \beta$ if $2m - 2\alpha - \beta \ge 0$ and the first term of the right-hand side appears in this case only. Put $c_{p+k} = d_k$ for $1 \le k \le q$. Let δ_j be a curve making one turn around c_k for $1 \le k \le p+q$. Then, for $1 \le k \le p$, $I(\delta_k)$ makes one turn around $-\bar{c}_k$ in the opposite direction. Hence it holds

$$\mathrm{Re}\left(\int_{I(\delta_k)} \phi_i \right) = \mathrm{Re}\left(\int_{\delta_k} \bar{\phi}_i \right) = \mathrm{Re}\left(\int_{\delta_k} \phi_i \right).$$

Now, $\mathrm{Re}(\int_{\delta_k} \phi_i) = 0$, $1 \le k \le p+q$, for $i = 1, 2$ if and only if

$$\int_{\delta_k} \psi_1 + \int_{\delta_k} I^*(\psi_1) = 0, \qquad 1 \le k \le p+q. \tag{3.5}$$

On the other hand, $\mathrm{Re}(\int_{\delta_k} \phi_3) = 0$, $1 \le k \le p+q$, if and only if

$$\mathrm{Im}\,\mathrm{Res}_{z=c_k}\left(AB\prod_{j=1}^{m}(z-b_j)(z+\overline{b}_j)/R\right) = 0, \qquad 1 \le k \le p+q. \tag{3.6}$$

If $c_k \ne a_i$ for all $1 \le i \le n$, then we put

$$R_j(c_k) = (1/w)^{(j-1)}(c_k)/(j-1)!. \tag{3.7}$$

On the other hand, if $c_j = a_i$ for some i, we put

$$R_k(c_j) = (1/w_i)^{2k}(0)/2k!, \tag{3.8}$$

where

$$w_i = \sqrt{\begin{aligned}&(a_1 - a_i + t^2)(\overline{a}_1 + a_i - t^2) * \cdots * (\overline{a}_{i-1} + a_i - t^2)(\overline{a}_i + a_i - t^2)\\ &*(a_{i+1} - a_i + t^2) * \cdots * (a_n - a_i + t^2)(\overline{a}_n + a_i - t^2).\end{aligned}}$$

Now, we can state the conditions under which the 1-form Φ has no real period around the endpoints.

Lemma 3.2. *The vector-valued* 1*-form* Φ *has no real period around the endpoints which are different from* (∞, ∞) *if and only if*

$$\begin{aligned}&\sum\sum_{j=1}^{\alpha_k}(B_{kj}R_j(c_k) - (-1)^j C_{kj}\overline{R_j(c_k)}) = 0, \quad \text{for } 1 \le k \le p,\\ &\sum_{j=1}^{\beta_k} D_{kj}(R_j(d_k) - (-1)^j R_j(d_k)) = 0, \quad 1 \le k \le q,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathrm{Re}(F_{k1}) &= 0, \qquad 1 \le k \le p \text{ if } \beta \text{ is even},\\ \mathrm{Im}(F_{k1}) &= 0, \qquad 1 \le k \le p \text{ if } \beta \text{ is odd}.\end{aligned} \tag{3.10}$$

If the endpoint p_r *is* (∞, ∞), *then* $p+q = r-1$ *and* $2m \ge 2p+q$. *The* 1*-form* Φ *has no real period around the endpoint* $p_r = (\infty, \infty)$ *if and only if*

$$\sum_{j=n-1}^{l} A_j(\widetilde{R}_{2+j-n}(0) - (-1)^j\overline{\widetilde{R}_{2+j-n}(0)}) = 0, \tag{3.11}$$

where

$$\widetilde{R}_j(0) = (1/\tilde{w})^{(j-1)}(0)/(j-1)! \quad \text{with } \tilde{w} = \sqrt{\prod_{i=1}^{n}(a_i\xi - 1)(\overline{a}_i\xi + 1)}.$$

Proof. Using (3.3), we get from (3.5), for $1 \le k \le p$,

$$\sum_{j=1}^{\alpha_k}(B_{kj}\,\mathrm{Res}_{z=c_k}(1/(z-c_k)^j w) - (-1)^j C_{kj}\overline{\mathrm{Res}_{z=c_k}\,(1/(z-c_k)^j w)}) = 0$$

and for $1 \le k \le q$,

$$\sum_{j=1}^{\beta_k} D_{kj}(\mathrm{Res}_{z=d_k}(1/(z-d_k)^j w) - (-1)^j\overline{\mathrm{Res}_{z=d_k}(1/(z-d_k)^j w)}) = 0.$$

If $c_j \neq a_i$ for all $1 \leq i \leq n$, we obtain (3.9). If $c_j = a_i$ for some i, we may take a local coordinate t near c_j such that $a_i - z = t^2$. Using (3.8), we also have (3.9) in this case. The conditions (3.6) are reduced to $\mathrm{Im}((ABF_{k1}) = 0$, $1 \leq k \leq p$, $\mathrm{Im}((ABG_{k1}) = 0$, $1 \leq k \leq q$. If β is even, AB is an imaginary number and G_{k1} are also imaginary numbers. Hence we get from the above $\mathrm{Re}(F_{ki}) = 0$. Similarly, if β is odd, we have $\mathrm{Im}(F_{ki}) = 0$. Now assume $p_r = (\infty, \infty)$. Set $z = 1/\xi$. Then we have

$$\psi_1 = -\sum_{j=n-1}^{l} A_j/(\xi^{2+j-n}\tilde{w}) + (\text{the holomorphic part near } p_r),$$

$$\psi_2 = -\sum_{j=n-1}^{l} (-1)^j \overline{A}_j/(\xi^{2+j-n}\tilde{w}) + (\text{the holomorphic part near } p_r).$$

Hence we obtain (3.11). From the conditions (3.6), we get

$$\mathrm{Im}(AB(F_{k1} - (-1)^\beta F_{k1}) = 0, \qquad \mathrm{Im}(ABG_{k1}) = 0.$$

But these hold without any assumption.

4. Preliminaries for the proof of the theorem

In this section, we will apply Lemmas 3.1 and 3.2 to elliptic Riemann surface and corresponding nonorientable surfaces $M = P_2 - \{\pi(\infty, \infty)\}$ of genus 2 with one end. In this case, a Weierstrass pair in Lemma 2.1 is given by

$$g = kA\prod_{j=1}^{m}(z - b_j)\prod_{i=1}^{2}(z - a_i)/w\prod_{j=1}^{m}(z + \overline{b}_j),$$

$$f = B\prod_{j=1}^{m}(z + \overline{b}_j)^2\prod_{i=1}^{2}(z + a_j)/w,$$

where $w = \sqrt{(1 - z^2)(1 - k^2z^2)}$, $|A| = 1$, AB is a pure imaginary number, $a_1 = 1$, $a_2 = a = 1/k$ or $-1/k$. Notice $\deg g = 2m + 2$. In the present case, we can put

$$\psi_1 = k\left(\prod_{j=1}^{m}(z + \overline{b}_j)^2(z + 1)(z + a)/w\right)dz = k\left(\sum_{j=1}^{2m+2} A_j z^j\right)dz/w.$$

To apply Lemma 3.1, we may take the following curves. $\beta_1(t) = (t, w(t))$, $-1 \leq t \leq 1$, $\beta_2(t) = (t, w(t))$, $1 \leq t \leq 1/k$. Then we have

$$I_1^j = \int_{\beta_1}(z^j/w)\,dz = \int_{-1}^{1}(t^j/w)\,dt,$$

$$I_2^j = \int_{\beta_2}(z^j/w)\,dz = \int_{1}^{1/k}(t^j/w)\,dt.$$

Remark that

$$\int_{-1}^{0}(t^{2j}/w)\,dt = \int_{0}^{1}(t^{2j}/w)\,dt \quad\text{and}\quad \int_{-1}^{0}(t^{2j+1}/w)\,dt = -\int_{0}^{1}(t^{2j+1}/w)\,dt.$$

Moreover all I_i^{2j} are real numbers. Now, as $l = 2m+2$, the relations in Lemma 3.1 are reduced to

$$\sum_{j=0}^{m+1} I_i^{2j} A_{2j} = 0. \tag{4.1}$$

On the other hand, Φ has no real period around the endpoint if and only if it holds (3.11), that is,

$$\sum_{h=0}^{m} A_{2h+1}\widetilde{R}_{2h+1} = 0, \tag{4.2}$$

where we use the fact that $\widetilde{R}_k = \widetilde{R}_k(0)$ are real numbers.

Let B_j be the elementary symmetric polynomials of b_j. Moreover we set

$$S_h = \sum_{i+j=h} B_i B_j, \qquad 0 \le h \le 2m.$$

Then we have

$$\begin{aligned}
&A_{2m+2} = S_0 = 1, \quad A_{2m+1} = \overline{S}_1 + (1+a), \\
&A_h = \overline{S}_{2m+2-h} + (1+a)\overline{S}_{2m+1-h} + a\overline{S}_{2m-h}, \qquad 2 \le h \le 2m, \\
&A_1 = (1+a)\overline{S}_{2m} + a\overline{S}_{2m-1}, \quad A_0 = a\overline{S}_{2m},
\end{aligned}$$

where $a = 1/k$ or $-1/k$ and k is the modulus of the elliptic Riemann surface. Thus, we have

Lemma 4.1. *The vector-valued* 1*-form* Φ *has no real period if and only if for* $i = 1, 2$,

$$\sum_{h=0}^{m}(I_i^{2m+2-2h} + aI_i^{2m-2h})S_{2h} + (1+a)\sum_{h=0}^{m-1} I_i^{2m-2h} S_{2h+1} = 0, \tag{4.3}$$

$$\sum_{h=0}^{m-1}(a\widetilde{R}_{2m-2h-1} + \widetilde{R}_{2m-2h+1})S_{2h+1} + (1+a)\sum_{h=0}^{m} \widetilde{R}_{2m-2h+1} S_{2h} = 0, \tag{4.4}$$

where

$$\begin{aligned}
&I_1^j = \int_0^l (t^j/w)\,dt, \quad I_2^j = \int_1^{1/k} (t^j/w)\,dt, \\
&\widetilde{R}_j = (1/\tilde{w})^{(j-1)}(0)/(j-1)!, \quad \tilde{w} = \sqrt{(\xi^2-1)(\xi-k^2)}.
\end{aligned}$$

The following formula concerning elliptic integrals is well known (*see, for example, the formulas* 17.1.4 *and* 17.1.5 *on p.* 589 *in* [1]).

$$(2j+3)I_i^{2j+4} - (2j+2)(a^2+1)I_i^{2j+2} + (2j+1)a^2 I_i^{2j} = 0, \qquad i = 1, 2. \tag{4.5}$$

5. A proof of the theorem

At first, we will show the part (1) of the theorem, that is, there is no complete minimal immersion of M into R^3 with total curvature $-4\pi(m+1)$, $m =$

$0, 1, 2$. At some calculations in the present sections, we use the computer algebra system REDUCE 3.2.

In the case $m = 0$, the equation (4.3) is reduced to

$$I_i^2 + aI_i^0 = 0, \qquad i = 1, 2. \tag{5.1}$$

From the fundamental formulas of elliptic integrals (see, for example, §3.1 of [6]), we have

$$I_1^0 = K(k), \quad I_2^0 = -\sqrt{-1}K(k') = -\sqrt{-1}K'(k), \tag{5.2}$$

$$I_i^2 = a^2(I_i^0 - E_i(k)), \tag{5.3}$$

where $k' = \sqrt{1-k^2}$,

$$E_1 = E(k) = \int_0^1 \sqrt{(1-k^2z^2)/(1-z^2)}dz, \tag{5.4}$$

$$\begin{aligned} E_2 = E(k) &= \int_1^{1/k} \sqrt{(1-k^2z^2)/(1-z^2)}dz \\ &= -\sqrt{-1}(K'(k) - E'(k)). \end{aligned} \tag{5.5}$$

By substituting (5.2), (5.3), (5.4) and (5.5) into (5.1), we obtain $(1+a)K = aE$, $aE' + K' = 0$. From this we get $KE' + K'E - KK' = 0$. This contradicts the formula

$$KE' + K'E - KK' = \pi/2. \tag{5.6}$$

In the case $m = 1$, the equation (4.3) is reduced to

$$I_i^4 + aI_i^2 + S_2(I_i^2 + aI_i^0) + (1+a)S_1I_i^2 = 0. \tag{5.7}$$

Put $b_1 = b$. Then $S_1 = 2b$, $S_2 = b^2$. Using (5.2), (5.3), (5.4), (5.5) and (4.5), we get

$$\begin{aligned} &3((a+1)K - aE)b^2 + 6a(a+1)(K-E)b \\ &\quad + a(2a^2+3a+1)K - a(2a^2+3a+2)E = 0, \end{aligned} \tag{5.8}$$

$$3(aE' + K')b^2 + 6a(a+1)E'b + a(2a^2+3a+2)E' - aK' = 0. \tag{5.9}$$

If these two equations have a common solution, it must be $b = -(a+1)/3$. Substituting this into (5.8) and (5.9), we obtain

$$(a^2 - a + 1)(aE - aK - K) = 0, \quad (a^2 - a + 1)(aE' + K') = 0.$$

Here we use the computer algebra system. These equations contradict to (5.6).

In this case $m = 2$, (4.3) is reduced to

$$A_1w^2 + 2A_2zw + A_3(z^2 + 2w) + 2A_4z + A_5 = 0, \tag{5.10}$$

$$B_1w^2 + 2B_2zw + B_3(z^2 + 2w) + 2B_4z + B_5 = 0, \tag{5.11}$$

where $z = B_1 = b_1 + b_2$, $w = B_2 = b_1b_2$ and $A_1 = I_1^2 + aI_1^0$, $A_2 = (1+a)I_1^2$, $A_3 = (I_1^4 + aI_1^2)$, $A_4 = (1+a)I_1^4$, $A_5 = I_1^6 + aI_1^4$, $B_1 = I_2^2 + aI_2^0$, $B_2 = (1+a)I_2^2$,

$B_3 = (I_2^4 + aI_2^2)$, $B_4 = (1+a)I_2^4$, $B_5 = I_2^6 + aI_2^4$. From the equations (5.10) and (5.11), we get

$$2F_1 zw + F_2(z^2+2w) + 2F_3 z + F_4 = 0, \tag{5.12}$$

where $F_i = A_{i+1}B_1 - A_1 B_{i+1}$. Hence we obtain, when $z \neq -2(a+1)/3$,

$$w = -(5(a+1)z^2+5(2a^2+a+2)z+(4a^2-a+4)(a+1))/(5(3z+2(a+1))). \tag{5.13}$$

Substituting this into (5.10) and (5.11), we obtain the following equation coming from symbolic algebra manipulation.

$$25G_4 z^4 + 50G_3 z^3 + 10G_2 z^2 + 20G_1 z + G_0 = 0, \tag{5.14}$$

where $G_4 = a^2 - a + 1$, $G_3 = 2a^3 - a^2 - a + 2$, $G_2 = 14a^4 + a^3 - 14a^2 + a + 14$, $G_1 = 4a^5 + 3a^4 - 4a^3 - 4a^2 + 3a + 4$, $G_0 = 16a^6 + 24a^5 - 7a^4 - 30a^3 + 7a^2 + 24a + 16)$. If $z = -2(a+1)/3$, substituting this into (5.12), we have $(K'E + KE' - KK')a^3(4a^2 - a + 4)(a+1)^2 = 0$. But, we have no real solution for this equation. Hence we get $z \neq -2(a+1)/3$.

Next, we will investigate the condition (4.4). In the case $m = 2$, this is reduced to

$$\begin{aligned}(a+1)R_1 w^2 + 2(aR_1 + R_3)wz + (1+a)R_3(z^2+2w)\\ + 2(aR_3 + R_5)z + (1+a)R_5 = 0.\end{aligned} \tag{5.15}$$

By calculation, we have $R_{2i} = 0$ and $R_1 = 1$, $R_3 = (1+a^2)/2$,

$$R_5 = (3a^4 + 2a^2 + 3)/2^3, \quad R_7 = (5a^6 3a^4 + 3a^2 + 5)/2^5,$$
$$R_9 = (35a^8 + 20a^6 + 18a^4 + 20a^2 + 35)/2^5.$$

Substituting (5.13) into (5.15), we obtain

$$100H_4 z^4 + 50H_3 z^3 + 5H_2 z^2 + 60H_1 z + H_0 = 0, \tag{5.16}$$

where $H_4 = 5a^2 - 8a + 5$, $H_3 = 23a^3 - 15a^2 - 15a + 23$, $H_2 = 223a^4 + 64a^3 - 318a^2 + 64a + 223$, $H_1 = 9a^5 + 11a^4 - 12a^3 - 12a^2 + 11a + 9$, $H_0 = 108a^6 + 232a^5 - 36a^4 - 320a^3 - 36a^2 + 232a + 108$. Thus the problem is to find a common solution for (5.14) and (5.16). Now we consider Sylvester's resultant of those equations, which is a determinant of order 8. The computation of the determinant is accomplished by the computer algebra system REDUCE 3.2. In fact the resultant of the equations is as follows

$$\begin{aligned}R = 390625(&3721a^{12} + 7442a^{11} + 8299a^{10} - 16958a^9\\ &- 14717a^8 - 1684a^7 + 48530a^6 - 1684a^5\\ &- 14717a^4 - 16958a^3 + 8299a^2 + 7442a + 3721)\\ &\cdot (4a^2 - a + 4)(a+1)^4(a-1)^8.\end{aligned}$$

We can show that there is no solution a for $R = 0$ which satisfies $|a| > 1$. Thus we obtain part (1) of the theorem.

In order to prove the part (2) of the theorem, when $m = 3$ we put $x = 1$, $y = b_1 + b_2 + b_3$, $z = b_1 b_2 + b_2 b_3 + b_3 b_1$, $w = b_1 b_2 b_3$. Then the equations

(4.3) for $i = 1, 2$ are rewritten respectively as

$$\begin{aligned} &A_1 w^2 + A_3 z^2 + A_5 y^2 + A_7 x^2 \\ &\qquad + 2(A_2 wz + A_3 wy + A_4 wx + A_4 yz + A_5 zx + A_6 yx) = 0, \\ &B_1 w^2 + B_3 z^2 + B_5 y^2 + B_7 x^2 \\ &\qquad + 2(B_2 wz + B_3 wy + B_4 wx + B_4 yz + B_5 zx + B_6 yx) = 0, \end{aligned}$$

where A_1, A_2, A_3, A_4, A_5 and B_1, B_2, B_3, B_4, B_5 are the same coefficients in (5.10) and (5.11), and $A_6 = (1+a)I_1^6$, $A_7 = I_1^8 + aI_1^6$, $B_6 = (1+a)I_2^6$, $B_7 = I_2^8 + aI_2^6$. The equation (4.4) is also written as

$$\begin{aligned} &C_1 w^2 + C_3 z^2 + C_5 y^2 + C_7 x^2 \\ &\qquad + 2(C_2 wz + C_3 wy + C_4 wx + C_4 yz + C_5 zx + C_6 yx) = 0. \end{aligned}$$

Though $x = 1$, we now assume that x takes any complex values. Then the above three equations give three quadrics in the 3-dimensional complex projective space $CP^3 = \{(w, z, y, x)\}$. Then it is evident that the intersection of the three quadrics is not empty. We will show that there is no point with coordinate $(w, z, y, 0)$ in the intersection. In fact, if we have such a point in the intersection, the three equations are reduced to

$$\begin{aligned} A_1 w^2 + A_3 z^2 + A_5 y^2 + 2(A_2 wz + A_3 wy + A_4 yz) &= 0, \\ B_1 w^2 + B_3 z^2 + B_5 y^2 + 2(B_2 wz + B_3 wy + B_4 yz) &= 0, \\ C_1 w^2 + C_3 z^2 + C_5 y^2 + 2(C_2 wz + C_3 wy + C_4 yz) &= 0. \end{aligned}$$

These equations have no solution with $y \neq 0$, because in this case the above equations are reduced to the equations (4.3) and (4.4) for $m = 2$. On the other hand, if $y = 0$, the equations are reduced to (4.3) and (4.4) for $m = 1$ or $m = 0$. Hence there are no solutions. Thus the intersection consists of points with coordinate $(w, x, y, 1)$.

Now it is evident that for $m \geq 4$, equations (4.3) and (4.4) have common solutions.

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References

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1965.
2. J. L. M. Barbosa and A. G. Colares, *Minimal surfaces in* R^3, Lecture Notes in Math., vol. 1195, Springer, Berlin, 1986.
3. A. A. de Barros, *Complete nonorientable minimal surfaces in* R^3 *with finite total curvature*, An. Acad. Brasil. Cien. **59** (1987), 141–143.
4. D. Hoffman and W. H. Meeks III, *Embedded minimal surfaces of finite topology*, Ann. of Math. **131** (1990), 1–34.
5. T. Ishihara, *Complete Möbius strips minimally immersed in* R^3, Proc. Amer. Math. Soc. **107** (1989), 803–806.
6. D. F. Lawden, *Elliptic functions and applications*, Springer-Verlag, New York, 1989.

7. W. Meeks, *The classification of complete minimal surfaces in* R^3 *with total curvature greater than* -8π, Duke Math. J. **48** (1981), 523–535.

8. M. E. Oliveira, *Some new examples of nonorientable minimal surfaces*, Proc. Amer. Math. Soc. **98** (1986), 626–635.

9. R. Osserman, *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) **82** (1964), 340–364.

10. G. Springer, *Introduction to Riemann surface*, Addison-Wesley, Reading, Mass., 1957.

11. K. Yang, *Meromorphic functions on a compact Riemann surface and associated complete minimal surfaces*, Proc. Amer. Math. Soc. **105** (1989), 706–711.

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